First-Order Logic

Part Two

Recap from Last Time

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects,
  - ***functions*** that map objects to one another, and
  - ***quantifiers*** that allow us to reason about many objects at once.

Some muggle is intelligent.

$$\exists m.\ (Muggle(m) \land Intelligent(m))$$

∃ is the ***existential quantifier*** and says "for some choice of m, the following is true."

"For any natural number *n*,
n is even if and only if n^2 is even"

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow Even(n^2)))$$

∀ is the ***universal quantifier*** and says "for any choice of *n*, the following is true."

“Some P is a Q”

translates as

$$\exists x.\ (P(x) \land Q(x))$$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$$\exists x.\ (P(x) \land Q(x))$$

If x is an example, it *must* have property P on top of property Q.

“All *P*'s are *Q*'s”

translates as

$$\forall x.\ (P(x) \to Q(x))$$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$$\forall x.\ (P(x) \to Q(x))$$

If *x* is a counterexample, it *must* have property *P* but not have property *Q*.

New Stuff!

The Aristotelian Forms

"All *A*s are *B*s"

$\forall x. (A(x) \to B(x))$

"Some *A*s are *B*s"

$\exists x. (A(x) \land B(x))$

"No *A*s are *B*s"

$\forall x. (A(x) \to \neg B(x))$

"Some *A*s aren't *B*s"

$\exists x. (A(x) \land \neg B(x))$

It is worth committing these patterns to memory. We'll be using them throughout the day and they form the backbone of many first-order logic translations.

The Art of Translation

Using the predicates

- *Person*(*p*), which states that *p* is a person, and
- *Loves*(*x*, *y*), which states that *x* loves *y*,

write a sentence in first-order logic that means “every person loves someone else.”

Answer at

https://pollev.com/cs103aut23

$$
\begin{aligned}
&\forall p.\ (Person(p) \to \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\quad\quad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

Using the predicates

- *Person*(p), which states that p is a person, and
- *Loves*(x, y), which states that x loves y,

write a sentence in first-order logic that means “there is a person that everyone else loves.”

Answer at

https://pollev.com/cs103aut23

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ (Person(q)\ \land\ p \neq q \rightarrow \\
&\qquad Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: “Every person loves someone else”

For every person…	$\forall p.\ (Person(p) \rightarrow$
… there is another person …	$\exists q.\ (Person(q) \land p \neq q \land$
… they love	$Loves(p, q)$
	$)$
	$)$

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: “There is someone everyone else loves.”

There is a person…	$\exists p.\ (Person(p)\ \land$
… that everyone else …	$\forall q.\ (Person(q)\ \land\ p \neq q \rightarrow$
… loves.	$Loves(q, p)$
	$)$
	$)$

For Comparison

For every person…

… there is another person …

… they love

$$\forall p.\ (Person(p) \to \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

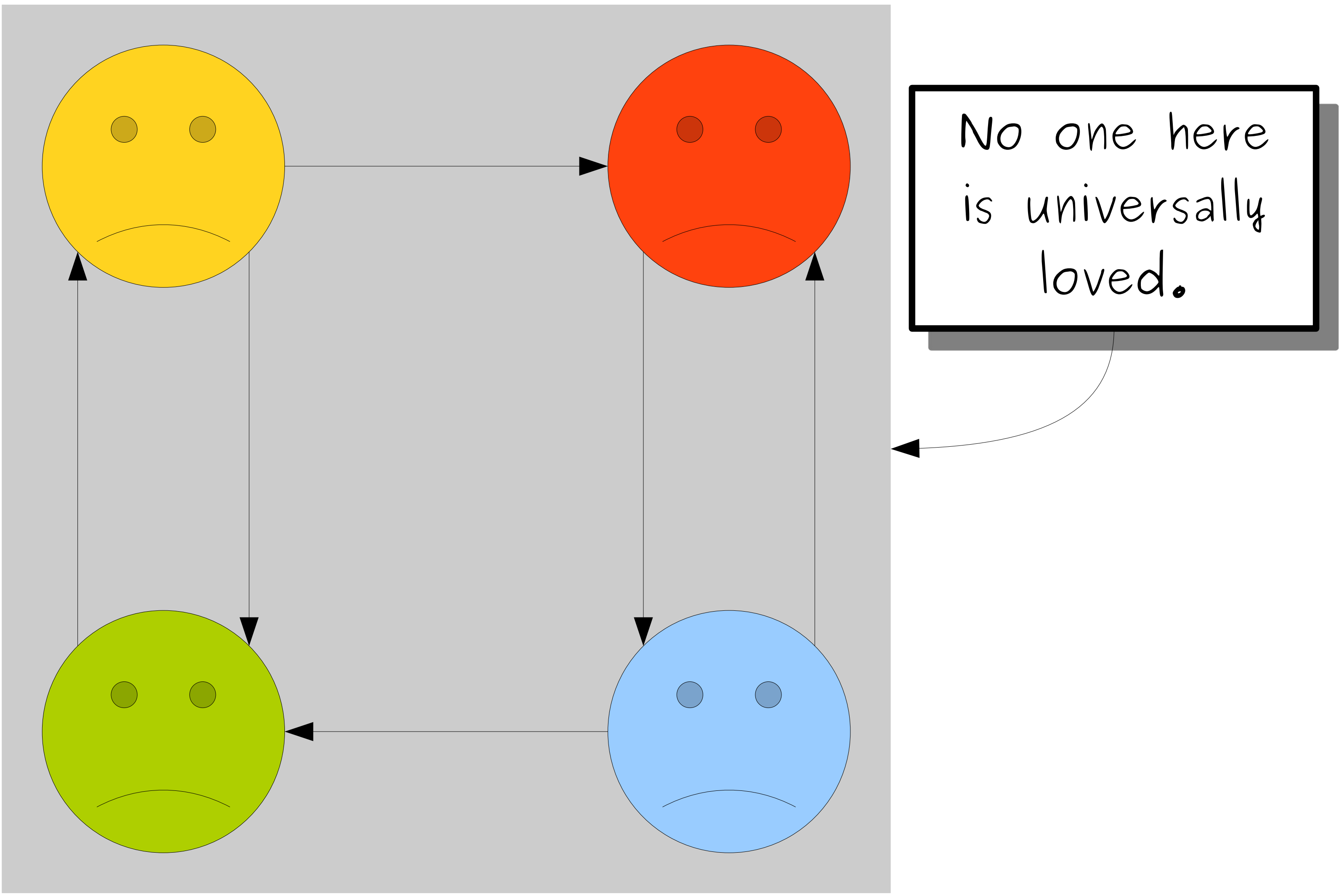
There is a person…

… that everyone else …

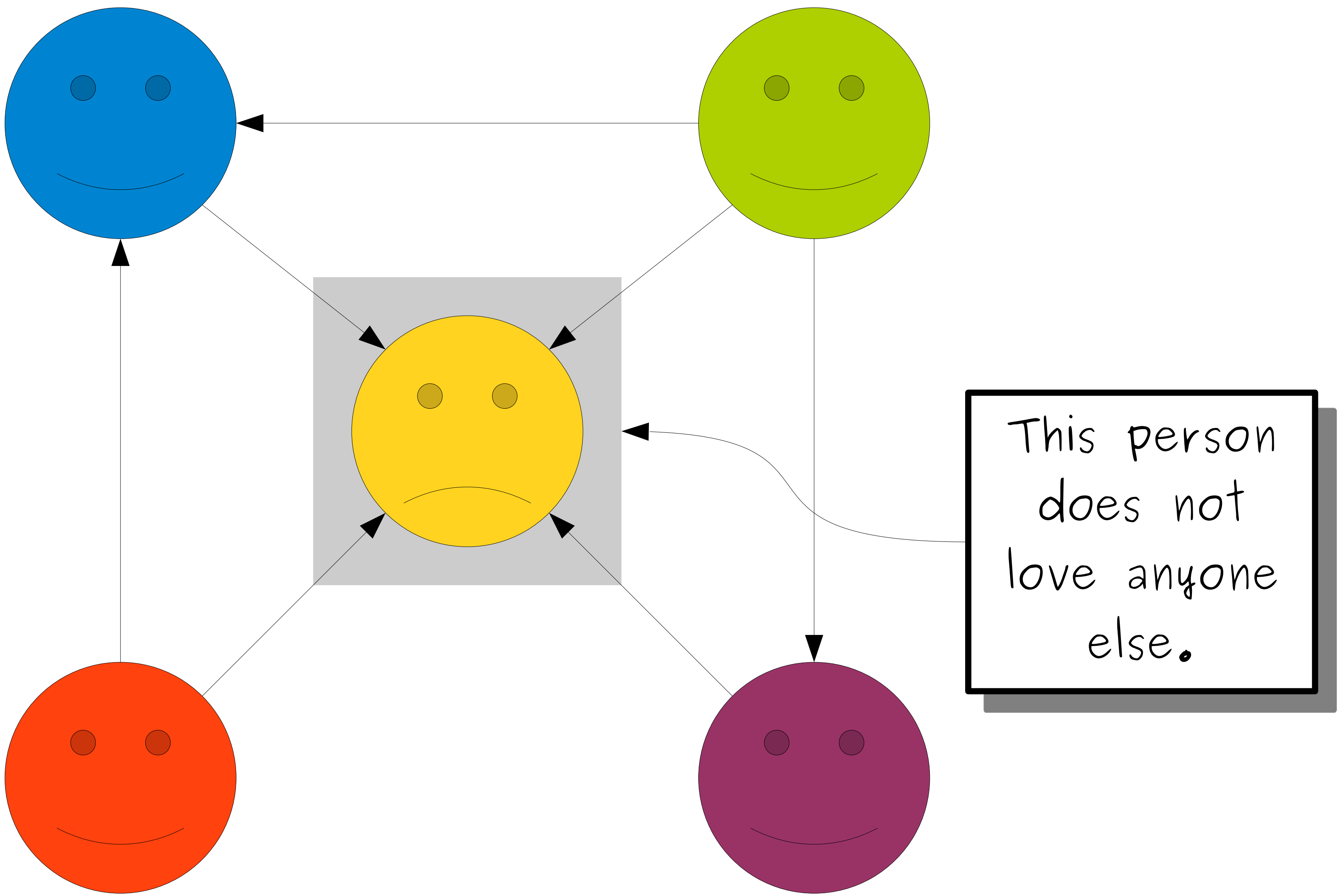
… loves.

$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \to Loves(q, p)))$$

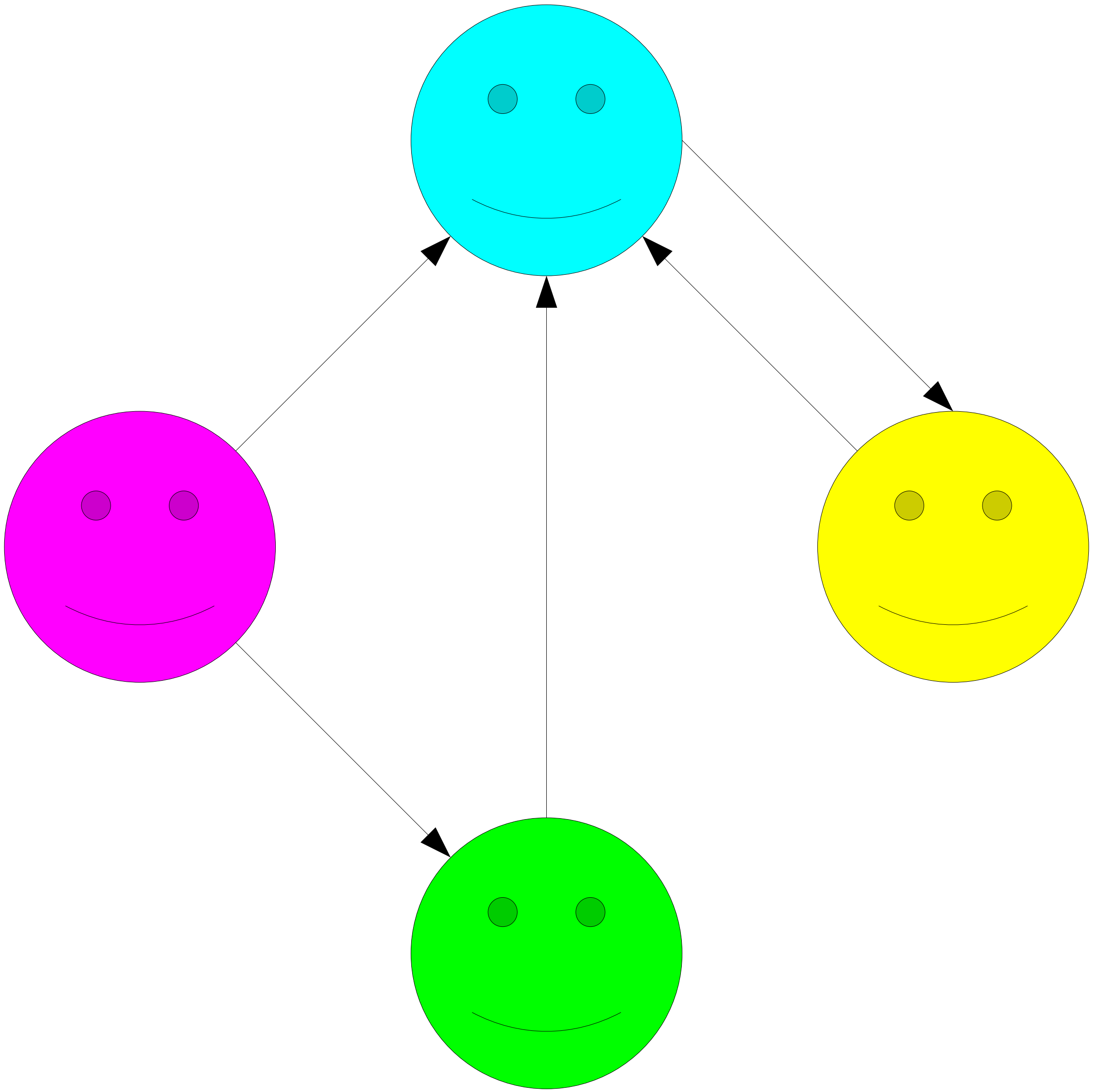
Every Person Loves Someone Else



There is Someone Everyone Else Loves



Every Person Loves Someone Else ***and*** There is Someone Everyone Else Loves



For every person…	$\forall p.\ (Person(p) \rightarrow$
… there is another person …	$\exists q.\ (Person(q) \land p \neq q \land$
… they love	$Loves(p, q)$
	$)$
	$)$
and	$\land$
There is a person…	$\exists p.\ (Person(p) \land$
… that everyone else …	$\forall q.\ (Person(q) \land p \neq q \rightarrow$
… loves.	$Loves(q, p)$
	$)$
	$)$

Quantifier Ordering

- The statement

  $$\forall x.\ \exists y.\ P(x, y)$$

  means "for any choice of x, there's some choice of y where $P(x, y)$ is true."
- The choice of y can be different every time and can depend on x.

Quantifier Ordering

- The statement

  $$\exists x.\ \forall y.\ P(x, y)$$

  means "there is some *x* where for any choice of *y*, we get that $P(x, y)$ is true."
- Since the inner part has to work for any choice of *y*, this places a lot of constraints on what *x* can be.

Order matters when mixing existential and universal quantifiers!

Time-Out for Announcements!

Problem Set Two

- Problem Set One was due today at 1:00PM.
  - You can extend the deadline to 1:00PM Saturday using one of your late days. As usual, no late submissions will be accepted beyond 1:00PM Saturday without prior approval.
- Problem Set Two goes out today. It's due next Friday at 1:00PM.
  - Explore first-order logic!
  - Expand your proofwriting toolkit!
- We have some online readings for this problem set.
  - Check out the ***Guide to Logic Translations*** for more on how to convert from English to FOL.
  - Check out the ***Guide to Negations*** for information about how to negate formulas.
  - Check out the ***First-Order Translation Checklist*** for details on how to check your work.

Back to CS103!

Set Translations

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means “the empty set exists.”

First-order logic doesn't have set operators or symbols “built in.” If we only have the predicates given above, how might we describe this?

$$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

Both of these translations are correct. Just like in propositional logic, there are many different equivalent ways of expressing the same statement in first-order logic.

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

Why can we switch which quantifier we're using here?

Mechanics: Negating Statements

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	There is an x where $P(x)$ is false.
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	For all objects x, $P(x)$ is false.
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	There is an x where $P(x)$ is true.
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	For all objects x, $P(x)$ is true.

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	$\forall x.\ P(x)$

Negating First-Order Statements

- Use the equivalences

  $$\neg\forall x.\ A \quad \textit{is equivalent to} \quad \exists x.\ \neg A$$

  $$\neg\exists x.\ A \quad \textit{is equivalent to} \quad \forall x.\ \neg A$$

  to negate quantifiers.
- Mechanically:
  - Push the negation across the quantifier.
  - Change the quantifier from ∀ to ∃ or vice-versa.
- Use techniques from propositional logic to negate connectives.

Taking a Negation

$$\forall x.\ \exists y.\ Loves(x, y)$$

("Everyone loves someone.")

$$\neg\forall x.\ \exists y.\ Loves(x, y)$$

$$\exists x.\ \neg\exists y.\ Loves(x, y)$$

$$\exists x.\ \forall y.\ \neg Loves(x, y)$$

("There's someone who doesn't love anyone.")

Two Useful Equivalences

- The following equivalences are useful when negating statements in first-order logic:

  $\neg(p \land q)$ *is equivalent to* $p \to \neg q$

  $\neg(p \to q)$ *is equivalent to* $p \land \neg q$

- These identities are useful when negating statements involving quantifiers.
  - $\land$ is used in existentially-quantified statements.
  - $\to$ is used in universally-quantified statements.
- When pushing negations across quantifiers, we *strongly recommend* using the above equivalences to keep $\to$ with $\forall$ and $\land$ with $\exists$.

Negating Quantifiers

- What is the negation of the following statement, which says "there is a cute puppy"?

$$\boldsymbol{\exists x.\ (Puppy(x) \land Cute(x))}$$

- We can obtain it as follows:

$$\neg\exists x.\ (Puppy(x) \land Cute(x))$$

$$\forall x.\ \neg(Puppy(x) \land Cute(x))$$

$$\forall x.\ (Puppy(x) \to \neg Cute(x))$$

- This says "no puppy is cute."
- Do you see why this is the negation of the original statement from both an intuitive and formal perspective?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

(“There is a set with no elements.”)

$$\neg\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$
$$\forall S.\ \neg(Set(S) \land \forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \to \neg\forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \to \exists x.\ \neg\neg(x \in S))$$
$$\forall S.\ (Set(S) \to \exists x.\ x \in S)$$

(“Every set contains at least one element.”)

Restricted Quantifiers

Quantifying Over Sets

- The notation

$$\forall x \in S.\ P(x)$$

means "for any element x of set S, $P(x)$ holds." (It's vacuously true if S is empty.)

- The notation

$$\exists x \in S.\ P(x)$$

means "there is an element x of set S where $P(x)$ holds." (It's false if S is empty.)

Quantifying Over Sets

- The syntax

$$\forall x \in S.\ P(x)$$

$$\exists x \in S.\ P(x)$$

is allowed for quantifying over sets.

- In CS103, feel free to use these restricted quantifiers, but please do not use variants of this syntax.
- For example, don't do things like this:

$$\forall x \text{ with } P(x).\ Q(x)$$

$$\forall y \text{ such that } P(y) \land Q(y).\ R(y).$$

$$\exists P(x).\ Q(x)$$

Expressing Uniqueness

Using the predicate

- *WayToFindOut*(w), which states that w is a way to find out,

write a sentence in first-order logic that means "there is only one way to find out."

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \land \\
&\quad \forall x.\ (\mathit{WayToFindOut}(x) \rightarrow x = w) \\
&)
\end{aligned}
$$

Expressing Uniqueness

- To express the idea that there is exactly one object with some property, we write that
  - there exists at least one object with that property, and that
  - there are no other objects with that property.
- You sometimes see a special "uniqueness quantifier" used to express this:

$$\exists! x.\ P(x)$$

- For the purposes of CS103, please do not use this quantifier. We want to give you more practice using the regular $\forall$ and $\exists$ quantifiers.

Next Time

- ***Functions***
  - How do we model transformations and pairings?
- ***First-Order Definitions***
  - Where does first-order logic come into all of this?
- ***Proofs with Definitions***
  - How does first-order logic interact with proofs?